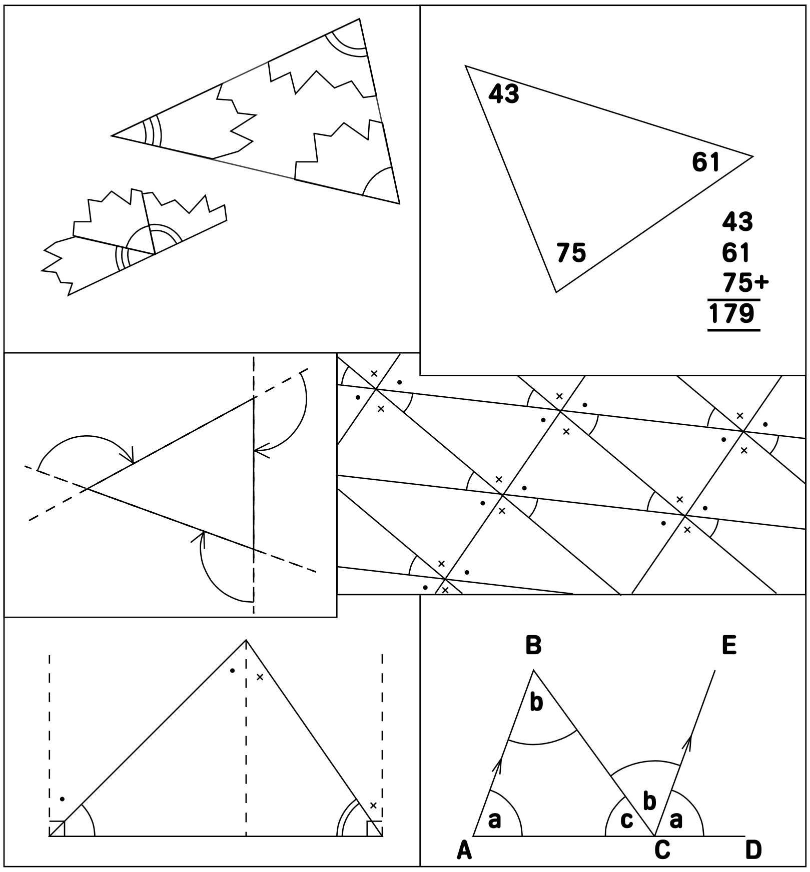
Can you prove it?

Developing concepts of proof in primary and secondary schools

Sue Waring



The Mathematical Association

Preface

This book arose out of a thesis based on research undertaken by the author, as part-time student and full-time teacher, although its origins go back much further.

The focus of the research was directly affected by the conflict between memories of pleasurable experiences when learning proofs as a schoolgirl, and the realisation, as a teacher in the early 1980s, that pupils were no longer exposed to these experiences. Not everyone has the same positive memories of school mathematics in the 1950s. For many pupils rote learning of some proofs and difficulties in writing proofs in an unnatural (to pupils) style were detrimental to their learning of mathematics. However, the wisdom of removing almost all reference to proof for all pupils during the curriculum changes of the 1960s has recently begun to be questioned.

The creation of a curriculum based on the perceived needs of pupils “from the bottom up” instead of the “top down” approach influenced by the universities was laudable but had both negative and positive impact. In replacing Euclidean geometry with the more accessible transformation geometry the new curriculum removed almost all traces of proof. So it no longer presented mathematics as an example of a unified logical system. The absence of proof impoverishes the mathematical experience of all pupils. The fact that methods of teaching it to date have not been appropriate for many pupils is not a valid reason for its neglect. Shifting the emphasis from the study of deductive, mainly formal, proofs to other more informal proofs makes ideas of proof accessible to more, and younger, pupils.

Although the disappearance of proof was a gradual process, which took place mainly during the 1970s, its absence was very noticeable to the author soon after returning to the classroom after a nine year break. It was pleasing to see that pupils were no longer required to rote learn proofs but it was disconcerting to realise that the ability to prove was no longer tested in “O-Level” mathematics examinations. Questions about proving theorems in geometry were replaced by calculations of angles and lengths; and proofs in other aspects of the curriculum were not examined. The effect of this shift in emphasis was clearly demonstrated to the author by the puzzled expressions on the faces of a group of “A-Level” students at this time, when they were asked why the sum of the angles of a triangle was 180°. This reinforced her belief that proof is a vital component of mathematics and should be part of pupils' mathematical experience, even if not explicitly tested.

The changes incurred by the advent of GCSE (1988) and the National Curriculum (1991) did not improve the situation. Proof was still accorded a very low priority when the research was started. Early drafts of the book expressed the purpose of the book as "raising the profile of proof". It was gratifying to be able to change this to "show how the curriculum can be delivered" in the light of recent changes to the (English) National Curriculum, which dramatically raise the profile of proof for all pupils.

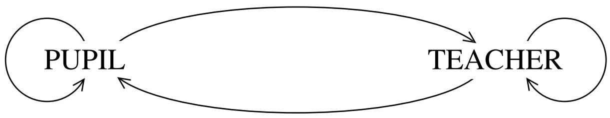
The research referred to above, which uses many of the proof activities in this book, provided evidence to support its hypotheses, namely:–

- experience in mathematical proof enhances appreciation of the need for proof;
- experience in mathematical proof improves ability to understand the nature of proof;
- experience in mathematical proof improves ability to construct a proof;
- teaching about proof through pattern is a viable alternative/ addition to teaching about proof by other methods.

This led the author to the conclusion that proof activities, especially those based on patterns, are a valuable feature of mathematics in primary and secondary schools. The first three hypotheses form the basis for the programme in the book, designed to teach proof in three phases:– "Learning about proof", "Learning to prove" and "Improving proof skills".

The prolonged neglect of explicit references to proof means that there are likely to be many younger teachers who have limited experience of proof in school mathematics; and many older teachers are more likely to have experience of learning and teaching only formal deductive proofs. The material and guidance in this book are intended to support both groups. In the first two phases, class discussions aim to develop explanations of new (to pupils) mathematics, and are generally accompanied by diagrams. The emphasis in proof activities is on proof as an explanation of the structure of pattern. The third phase uses a more formal approach to proof for older, able pupils.

The question "Can you prove it?", which forms the subtitle of this book, is implicit at all stages of learning and teaching about proof. Initially it is asked by the teacher of pupils in order to raise their awareness of the existence of proof. At an early stage pupils are encouraged to ask the teacher to prove new claims as they learn to expect that justification is almost always possible. In preparing to answer these questions teachers will need to be able to prove to themselves all the mathematics they teach to pupils. Finally, an important objective of teaching proof is that pupils begin to ask themselves such questions. The following diagram would seem to summarise the directions for this question:–



The author

I began teaching in a girls' grammar school in 1966. A career break of almost nine years (1969-77) to raise two daughters created opportunities to teach part-time in both the primary and tertiary sectors. Return to part-time teaching in secondary schools provided experience in girls', boys', and mixed selective and comprehensive schools. In 1991 I embarked on part-time study for an M.Ed. in Mathematical Education, followed by research for a doctoral thesis completed in 1997, both at the University of Leeds.

Acknowledgements

I wish to thank the following people for their assistance:–

my husband, Eric, for his moral support and practical help in many ways;

Anthony Orton and Tom Roper for their guidance during the preparation of my thesis;

Peter Bailey, Bill Richardson and others of the Mathematical Association, for helpful comments and support in preparing the manuscript for publication; and finally

the pupils I have taught, especially those at Queen Elizabeth's High School, Gainsborough, who participated in the research.

CHAPTER 1

Developing concepts of mathematical proof in school

This introductory chapter makes a strong case for developing ideas of proof in school mathematics. The development of pupil understanding of proof is described in six levels. A programme for teaching proof in three phases, from introducing the notion of proof to young children to enabling some older pupils to read and write more formal proof, provides –

"Proof-based mathematics for all pupils."

Introduction

Why teach proof?

"Proof is at the heart of mathematics" claims Anderson (1996, p 33). The author wholeheartedly supports this claim and, like some other mathematics educators, is concerned that in recent years proof has become almost extinct in school mathematics in England.

However significant changes proposed in the new version (July 1999) of the (English) National Curriculum for mathematics herald a welcome raising of the profile of proof in both primary and secondary schools. Indeed there is an explicit reference to pupils explaining their reasoning in the introduction to Key Stage 1 (p 37), and later a recommendation that in the "Shape, space and measures" component pupils should be taught to "use .... explanation skills as a foundation for geometrical reasoning and proof in later key stages." (p 39) This is a dramatic improvement on the 1994 proposals in which the first reference to developing skills of mathematical reasoning was in the introduction to Key Stages 3 and 4 and the only explicit reference to proof was as extension material.

Proof Levels

The following framework describes the development of proof concepts beginning with an appreciation of the need for proof, then an understanding of the nature of proof, and finally pupils' competence in constructing proofs. Thus the framework begins at **Proof Level 0**, which applies to pupils who have no appreciation of the notion of proof, and continues through five more levels to include pupils who understand the generalised nature of proof and its rôle in justifying conjectures, and who are also able to construct proofs in a variety of contexts.

Pupils who use "naive empiricism" (Balacheff, 1988) are at **Proof Level 1.** They recognise the existence of proof but do not appreciate its generalised nature. **Proof Level 2** represents a transitional stage between these two. Some pupils at this stage replace the few particular cases checked at Proof Level 1 with a greater number of particular cases, which are either more varied, eg. use larger numbers, or are randomly selected; others use a generic example to represent a class (Balacheff, 1988). Pupils who appreciate the generalised nature of proof are at **Proof Level 3.** They can follow a short chain of deductive reasoning but are not necessarily able to construct proofs. They can, however, distinguish between proof and practical demonstration. At **Proof Level 4** pupils can construct proofs in a limited range of contexts, including those which are familiar, and/or those which give rise to thought processes like “intuitive conviction” (Fischbein, 1982) or “transformational reasoning” (Simon, 1996), which a pupil may be able to express informally. Pupils at **Proof Level 5** have a deeper understanding of the nature and rôle of proof, and can construct proofs in a variety of contexts, possibly using some degree of formal language, where appropriate. All six levels, including a sub-division of Proof Level 2, are outlined below.

Proof Level 0: Pupils are ignorant of the necessity for, or even existence of, proof;

Proof Level 1: Pupils are aware of the notion of proof but consider that checking a few special cases is sufficient as proof;

Proof Level 2: Pupils are aware that checking a few cases is not tantamount to proof but are satisfied that either

a) checking for more varied and/or randomly selected examples is proof;

or b) using a generic example forms a proof for a class;

Proof Level 3: Pupils are aware of the need for a generalised proof and, although unable to construct a valid proof unaided, are likely to be able to understand the creation of a proof at an appropriate level of difficulty;

Proof Level 4: Pupils are aware of the need for, and can understand the creation of, generalised proofs and are also able to construct such proofs in a limited number of, probably familiar, contexts;

Proof Level 5: Pupils are aware of the need for a generalised proof, can understand the creation of some formal proofs, and are able to construct proofs in a variety of contexts, including some unfamiliar.

CHAPTER 2

First Phase – Teaching and learning about proof

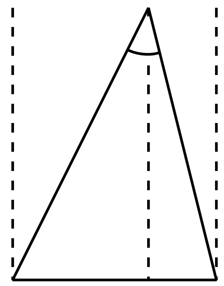
During this introductory phase of learning about proof, logical reasoning is used wherever possible to introduced new mathematics, and some familiar ideas are also justified through Proof Discussions. Proof Activities produce patterns and relationships whose justification involves use of oral explanation skills, as pupils are challenged to –

"Convince a friend!"

Introduction

The main purpose of teaching proof is that, as initial ideas about mathematical proof are introduced to pupils, they learn that mathematical statements can, and whenever possible should, be proved or explained. During this first phase pupils attain at least Proof Level 1. As they learn about the existence of proof through examples of proofs in the classroom, many will realise that checking a statement for one special case is insufficient as proof, and that either a special case should be treated as a generic example of a general statement or that more, varied or randomly selected, cases are considered. In other words many pupils can be expected to attain Proof Level 2.

The emphasis at this stage is on justifying number properties and patterns through generic examples ("action proofs" with diagrams) and geometry through transformational reasoning. However, able pupils will be capable of understanding the use of simple algebraic and geometric arguments as part of generalised proofs, and so may attain Proof Level 3. Repeated exposure to proofs or explanations will show pupils that new mathematical knowledge can be derived from familiar facts and properties, whatever their level of understanding of the detail of the proofs themselves. The emphasis here is on teaching an awareness of the existence of proof. Since this cannot be done in isolation, it may be that, during this phase, some pupils will only have a very superficial understanding of the essence of the arguments.



Probability

The tension between pupils' intuitive ideas about probability and correct understanding is familiar to teachers. The discussion here attempts to reconcile the discrepancy between pupils' estimate of a probability and its correct value, by both experimental observation and reasoned argument.

One event

Once pupils have understood the concept of probability and realise the advantages of assigning numerical values they can consider the simple event of tossing a coin. Class discussion about possible outcomes and associated probabilities should include "equally likely" ideas and then an estimate that the probability of (say) heads is a half. The reason behind this need not be explained to the pupils at this stage. Instead pupils should decide how many times they need to toss a coin in order to find the probability by experiment, and then carry out the experiment. Class discussion of the results would seem to confirm the result of 0.5, but at this point it is still a conjecture. Consideration of theoretical probability, as one way to get heads compared with two different outcomes, then proves the result.

Two events

As pupils first encounter the various outcomes when two coins are tossed they have no difficulty in identifying three different outcomes, namely, two heads, one head or no heads (or equivalents). Begin a class discussion about this by asking pupils for the possible outcomes and listing them on the board. Then consider the likelihood of each and how the probabilities might be quantified; or if pupils are familiar with the use of fractions for this, ask them to suggest values for the probabilities of the outcomes listed on the board. It is highly likely that at least one pupil will suggest that the probability of both heads (or any of the listed outcomes) is one third, and that no pupil will know the correct value.

At this point the need for experimental evidence is appreciated by the pupils and they should be given the opportunity for this and told to keep a tally for each of the three different outcomes. They should collect results until most realise that the original conjecture is incorrect. During the subsequent discussion several results could be combined to calculate the experimental probability of two heads. Pupils will modify their conjecture to $p = \frac{1}{4}$. The fact that this is still a conjecture should be emphasised, and the need for proof highlighted.

A theoretical proof can be developed by using a possibility space diagram to identify the four equally likely outcomes, along with the definition of expected theoretical probability, thus:–

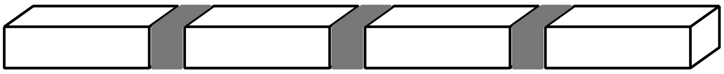
		1st coin	
2nd	H	HH	HT
coin	T	TH	TT

$$\text{Expected probability} = \frac{\text{Number of successful outcomes}}{\text{Number of equally likely different outcomes}}$$

This examination of all cases to confirm the correct result is a simple example of a proof by exhaustion.

Pupil Challenge

Bricklaying



Bricks are laid in lines with cement to join them.
How many joins for a line of bricks?

Find out for some short lines.
Look for a pattern.
Try to describe it.
Can you explain it?

Convince a friend

This is an example of a single operation (subtract one) number relationship, set in a visual real-life context. The number of bricks is deliberately not specified, in order to encourage pupils to think in general terms rather than about particular cases. It is expected that the class teacher will introduce this activity by explaining how to interpret the open numerical question, and asking pupils why they are advised to start with short lines. Pupils need not have a copy of the worksheet.

Lesson Outline (20-30 minutes)

Bricklaying

1. Explain the task. Discuss modelling.
2. Discuss how to start.
3. Provide materials and tell pupils to solve problem for 1, 2, and 3 bricks.
4. Ask pupils what they have found out. Record in table on board.
5. Explain that there is insufficient evidence. Pupils to continue collecting results.
6. Ask volunteers to add results to table.
7. Ask pupils to describe pattern.
8. Justify need for explanation. Ask pupils for theirs.

(9. Written reports only if appropriate.)

Lesson Plan

1. **Explain the task** to the pupils and clarify that they will need to choose the numbers of bricks. If the worksheet is used with primary pupils, teachers will need to check the instructions, and re-word them if necessary. **Discuss** with pupils what **materials/ apparatus** they could use to model this problem.

   One way is plastic bricks joined with plasticine, but there are equally good alternatives, such as drawing on squared paper.

2. **Discuss** with pupils how they will **start** the investigation.

   If this is the pupils' first experience in looking for number patterns they will need to realise the advantages of starting with one brick and then adding one at a time. Try to elicit this suggestion from pupils.

3. Organise pupils to work in pairs or small groups. **Provide materials/apparatus** or squared paper. Tell pupils to **find how many joins for up to three bricks**, by making models or drawing.

   This will not take very long. Allow a few pupils to consider four bricks if they finish early, but discourage them from guessing on the basis of continuing the number pattern.

4. **Stop practical work** and and **ask pupils what they have found** out. Allow one or two to tell the class their results. Ask them how they could **record this.**

   Try to elicit the suggestion of tabulating and complete the table on the board below with results from a different pupil.

Number of bricks	Number of joins
1	0
2	1
3	2

5. **Explain that that there is insufficient evidence** here to be able to make reliable predictions or draw safe conclusions. Pupils should **continue** practical investigation, and record their results.

   At this stage a few pupils may feel sure they know the answer. Tell them they can either collect more results to confirm what they know or consider how to explain why they know.

6. **Stop practical work** when most pupils have results for five or six bricks. Ask for **volunteers** to **add** their **results to the table** on the board. Ask some others if they agree with the values.

There are unlikely to be problems at this stage but it is important to resolve any that do occur until all pupils accept the values on the board.

7. Ask **pupils** if they can **describe how the number of joins is related to the number of bricks.**

   The most likely response is "one less," but some pupils may have other ways of expressing this. A few pupils may be able to express it algebraically as $j = b - 1$ where j and b represent the number of joins and bricks respectively.

8. **Explain** that even though this rule works for the lines of bricks made/drawn so far, **we need to know why** it works before we can assume it can be applied to longer lines.

   Ask pupils if they can explain why the rule should always work.

   Responses should be along the lines indicated below:–

   - The first brick on its own does not need any cement/have a join. The number of bricks with joins is one less than the total number of bricks. This is the same as the number of joins.

   or • In laying a line of bricks there is no need for cement on the last brick because there is not a new brick to join on. Counting the number of joins is the same as counting all but the last brick. So the number of joins is the number before the number of bricks.

   Ask several pupils for their version. Encourage full precise responses and always ask other pupils if they can suggest improvements. Although pupils may find this embarrassing at first, it is important that they learn to share the development of proofs. They also learn that full explanations are necessary and so partial answers need to be improved.

9. If a **written report** is required it need only include a copy of the table of results, a sentence like "The number of joins is one less than the number of bricks" and an explanation similar to those above. Most pupils working at this level will find writing this difficult. It is not essential for learning about proof.

   For many pupils, especially younger ones, written reports might leave them with a negative feeling about this work. A positive interest in learning about proof is far more important at this stage than writing about it.

CHAPTER 3

Phase 2 – Teaching and learning to prove

Pupils realise that, except for a few accepted truths (axioms), all mathematical statements can be subjected to proof which is independent of particular cases. Proof discussions are therefore more substantial in this second phase. As pupils attempt to explain the results of Proof Activities they begin to write short chains of deductive reasoning, as they are challenged to –

*"**Convince a penfriend!**"*

Introduction

The primary objectives during this phase are that pupils should develop a firm grasp of the generalised nature of proof (Proof Level 3); and that they learn to write simple, but generalised, proofs (Proof Level 4) independently if possible. Discussions about proving mathematical statements and also results of Proof Activities should become an integral part of the mathematical experiences of pupils, and not just isolated, occasional incidents in the classroom. In such a questioning environment pupils should adopt a questioning attitude to mathematics, and also develop a willingness to search for their own answers.

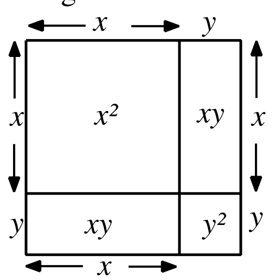
Algebra

Although none of the proof activities during this phase depend entirely on the use of algebra for completion of a satisfactory proof, most use the language of algebra to express relationships after they have been explained in words. Here are two basic algebraic relationships which lend themselves to a diagrammatic approach, namely the product of some binomials as shown below.

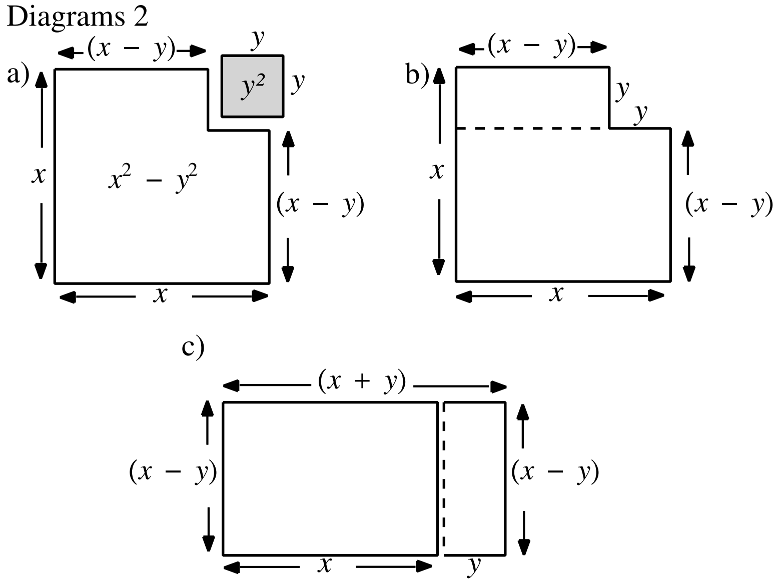
Multiplying binomials

While the expansion of $(x + y)^2$, shown in Diagram 1, is likely to be accessible in this form to many pupils, a similar diagram for $(x - y)^2$ is less helpful because of the difficulty of illustrating $-y$. For the same reason the expansion of $(x - y)(x + y)$ is not attempted, but instead its converse, the factorisation of the difference of two squares, is readily illustrated in Diagrams 2.

Diagram 1



Diagrams 2



CHAPTER 4

Phase 3 – Improving proof skills

Pupils in this last phase are secure in their knowledge and understanding of the generalised nature of proof, and their ability to write short proofs. In Proof Discussions they are exposed to traditional proofs in algebra and geometry. Proof Activities lead to pupils writing their own proofs of the results of more complex investigations. Pupils become familiar with some of the language of proof and can use a wider repertoire of proof methods. They are thus progressing –

towards formal proof.

Introduction

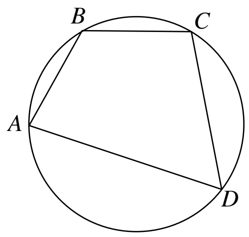
Pupils who acquire competence in writing proofs before examinations at sixteen are likely to benefit from some of the Proof Discussions and Proof Activities described in this chapter.

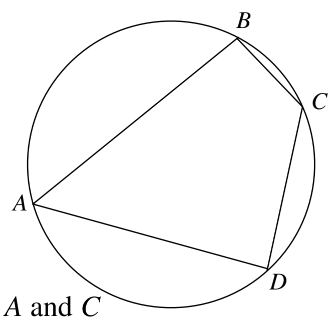
During this phase pupils are exposed to a wider range of proof types and contexts, and to more complex proofs, so that they can broaden and deepen their understanding of proof *per se*. They also practise creating a wider variety of proofs. Those who succeed in this will have attained Proof Level 5. It is also hoped that pupils acquire a deeper understanding of proof as an integral part of mathematics, and become conscious of the "wholeness" or unity of mathematics as opposed to an apparent discreteness which might be inferred from the approach in some current mathematics courses.

Pupil Challenge

Opposite angles in a cyclic quadrilateral

A cyclic quadrilateral is a quadrilateral with its vertices on the circumference of a circle.





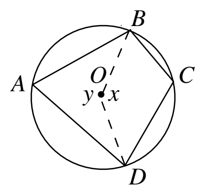
In each quadrilateral measure angles *A* and *C* carefully and write the size in each angle.

Repeat for angles *B* and *D*.

Describe what you notice about each pair of angles.

Use the diagram on the right and appropriate angle facts to explain why the result will always occur for opposite angles of any cyclic quadrilateral.

(*A*, *B*, *C*, and *D* are on the circumference of a circle centre *O*.)



Pupil responses

In the test nearly two thirds of each group noticed angle sums of 180° and a few identified equal angle sums. However, the only two proofs were from able pupils in the experimental group. One used an informal style and in the other the pupil had omitted reasons for the facts he used. One boy wrote, “In diagram 3 (on the question paper) the angles $x + y = 360°$; because $½x = a$ and $½y = c$, therefore $a + c = ½x + y$ which = 180°” and the girl wrote, “The angle in the centre of a circle is *O*. Angles of lines coming away from this will be half of it. Therefore *a* will be half of *x* and *c* will be half of *y*. $x + y$ = “*O*” = 360°. If they (*a* and *c*) are half of two numbers whose sum is 360° they must join together to equal 180°.” This last response was discussed with the girl during an interview and she was surprised that what she had written still made sense to her and even more surprised that it was essentially correct. Her comments in the interview were “I didn't think I got that right. I could remember bits of it and I had to try to piece it together to suit me.” She had clearly reached the stage of being able to construct her own proof from related mathematical facts, but still was unsure about what she produced.

One boy interviewed, who had not observed that opposite angles of a cyclic quadrilateral are supplementary, was asked how he might compare his pairs of values and suggested finding differences saying, “I always find differences first, and then sums.” He quickly recognised the pattern and was led to an understanding of the result. He commented, “I should have looked at it more carefully.” The remaining pupils interviewed were readily led through a proof based on the diagram given in the test, even though they had been unable to do this unaided. This suggests that if the proof is developed with the pupils through discussion it is not beyond their understanding.

CHAPTER 5

Proof in sixth forms and colleges

Three types of students with little, or no, prior knowledge of mathematical proof are exposed to notions of proof. Those needing help with basic numeracy are intrigued by the reasons behind some elementary mathematics. A general course about proof for non-mathematicians is outlined, and activities involving proof are suggested for students at the beginning of an A-Level Mathematics course. Thus at the post-sixteen phase this chapter offers ideas about –

proof for all.

Introduction

Since it seems likely that for some years to come some pupils will enter sixth forms and colleges with little or no experience of proof, it would seem appropriate to consider how to introduce older students to the notion of proof. For those studying A-level mathematics it is strongly recommended that this should be effected early in their course so that they learn to appreciate that proof is an intrinsic part of mathematics. It may also be possible to offer students who have elected to study other subjects a course of proof in mathematics as part of a general studies course, at a level appropriate to their mathematical ability. It is also possible that some older students who lack skills in numeracy might benefit from a proof-based approach.

All these groups could benefit by exposure to some of the proof discussions and activities already suggested for younger pupils. Those aiming to improve levels of basic numeracy could usefully consider some of the elementary properties of number. Students studying A-level mathematics, and some following a general course, could be introduced to the idea that mathematics is an example of a logical system with proof at its core. Those studying A-level mathematics should be enabled, through discussion, to develop proofs for all the new ideas they are taught, although these are not discussed here. They should also be given opportunities to construct proofs for themselves.